1. Prove or disprove

$(z \in \mathbb{C})$ $\lim_{z\to 0} z \sin(\frac{1}{z}) = 0$ Start at 10:35

2. Define $f(z) = \begin{cases} \frac{(\bar{z}^2)}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$ jxwang@math.cuhk.edu.hk

Show that (1) $f(z)$ satisfies Cauchy-Riemann eqts. at $z_0 = (0,0)$.

(2) if $z_0 = 0$, then $\frac{\Delta w}{\Delta z} = 1$ at each nonzero z along the real or imaginary axis.

(3) if $z_0 = 0$. then $\frac{\Delta w}{\Delta z} = -1$ at each nonzero $z = (\Delta x, \Delta x)$ on the $\Delta y = \Delta x$ line.

Conclude $f'(0,0)$ does not exist.

$(\Delta w := f(\underbrace{z_0 + \Delta z}_{z}) - f(z_0))$

1. Prove or disprove

$(z \in \mathbb{C})$ $\lim_{z\to 0} z \sin(\frac{1}{z}) = 0$

Real case $(z \in \mathbb{R})$

$$\sin(\tfrac{1}{z}) \in [-1,1]$$

then $$|z\sin(\tfrac{1}{z})| \overset{①}{\leq} |z|$$

$$0 \leq \lim_{z\to 0} |z\sin(\tfrac{1}{z})| \overset{②}{\leq} \lim_{z\to 0} |z| \overset{③}{=} 0$$

$\Rightarrow \lim_{z\to 0} z \sin(\frac{1}{z}) \overset{④}{=} 0$

Complex case ($z \in \mathbb{C}$)

Check ③ : $\lim_{z\to 0} |z| = 0$

OR < by $\varepsilon - \delta$ argument
Use the fact that
if $\lim_{z\to z_0} f(z) = w_0$, then $\lim_{z\to z_0} |f(z)| = |w_0|$
$z_0 = 0$. $f(z) = z$, $w_0 = 0$

Ex 7 in section 19 on page 55.

① : This is false.

Claim : range of $\sin(\frac{1}{z}) = \mathbb{C}$

Indeed, range of $\frac{1}{z} = \mathbb{C} \setminus \{0\}$

range of $\sin z = \mathbb{C}$

$\Rightarrow$ range of $\sin(\frac{1}{z}) = \mathbb{C}$

since $\sin(\frac{1}{z}) = 0$ on

$\{z = \frac{1}{2k\pi}\}$ for $k \in \mathbb{Z}\setminus\{0\}$.

$\sin(\frac{1}{z})$ is unbodded.

Disprove : compute as $z \to 0$ from positive imaginary axis.

i.e. $z = ai$ for some $0 < a \in \mathbb{R}$ s.t. $a \to 0$

then $\lim_{z\to 0} z \cdot \sin(\frac{1}{z}) = \lim_{a\to 0} ai \cdot \sin(\frac{1}{ai})$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$= \lim_{a\to 0} ai \, \frac{e^{\frac{1}{a}} - e^{-\frac{1}{a}}}{2i}$$

$$= \lim_{a\to 0} a \cdot \frac{e^{\frac{1}{a}} - e^{-\frac{1}{a}}}{2}$$

$$= \frac{1}{2} \lim_{a\to 0} \frac{e^{\frac{1}{a}}}{1/a}$$

$$\overset{\text{L'Hopital}}{=} \frac{1}{2} \lim_{a\to 0} \frac{-\frac{1}{a^2} e^{\frac{1}{a}}}{-\frac{1}{a^2}}$$

$$= \lim_{a\to 0} e^{\frac{1}{a}}$$

$$= \infty$$

The limit DNE.

Sequeezed THM (Real)

Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences in $\mathbb{R}$, and

$\lim_{n\to\infty} y_n = l$, $\lim_{n\to\infty} z_n = l$.

Suppose $\forall n \in \mathbb{N}$,

$$y_n \le x_n \le z_n$$

then $\lim_{n\to\infty} x_n = l$.

Sequeezed THM (Complex)

Let $\{a_n\}$ in $\mathbb{R}$ be a positive seq. s.t. $a_n \to 0$ as $n\to\infty$.

Let $\{z_n\}$ be a seq. in $\mathbb{C}$. Suppose $\forall n \in \mathbb{N}$,

$$|z_n| \le a_n$$

then $z_n \to 0$ as $n \to \infty$.

Pf: $\forall n \in \mathbb{N}$: $|z_n| \le a_n = |a_n|$

$\forall \varepsilon > 0$, $\exists N$ s.t. $|a_n| < \varepsilon$, $n > N$.

$\Rightarrow$ $|z_n| < \varepsilon$

$$\lim_{n\to\infty} |z_n| = 0$$

2. $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \ne 0 \\ 0, & z = 0 \end{cases}$

$u_x = v_y$, $u_y = -v_x$, $z = x+iy$

$f(z) = u(x,y) + i v(x,y)$

(1) When $z \ne 0$

$$f(z) = \underbrace{\frac{x^3 - 3xy^2}{x^2+y^2}}_{u(x,y)} + i\underbrace{\frac{y^3 - 3x^2y}{x^2+y^2}}_{v(x,y)}$$

$$u_x(x_0,y_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}$$

Thus, $u_x(0,0) = \lim_{\Delta x \to 0} \dfrac{u(\Delta x, 0) - u(0,0)}{\Delta x}$ $=0$ since $f(0) = 0$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$u_y(0,0) = 0$

$v_x(0,0) = 0$

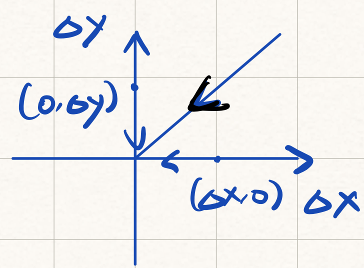
$v_y(0,0) = 1$

$\Rightarrow u_x = v_y$, $u_y = -v_x$

(2) Given $z_0 = 0$, $z = (x, y)$ ordered pair.

$$:= x + iy$$

$$\Delta z = z - 0 = (\Delta x, \Delta y) = (x, y)$$



$$\Delta z = (\Delta x, 0) \quad \text{or} \quad \Delta z = (0, \Delta y)$$

$$\frac{\Delta w}{\Delta z} = \frac{\;\frac{\overline{\Delta z}^2}{\Delta z}\;}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2 = 1$$

(3). $\Delta z = (\Delta x, \Delta x) = re^{i\frac{\pi}{4}}$ some $r \in \mathbb{R}$.

$$\frac{\Delta w}{\Delta z} = \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2 = \left(\frac{re^{-i\frac{\pi}{4}}}{re^{i\frac{\pi}{4}}}\right)^2 = e^{-\pi i} = -1$$

$\Delta z = (\Delta x, \Delta x)$, $\frac{\Delta w}{\Delta z} = -1$

$\Delta z = (\Delta x, 0)$, $\frac{\Delta w}{\Delta z} = 1$

$\Delta z = (0, \Delta y)$, $\frac{\Delta w}{\Delta z} = 1$.

$\Rightarrow$ $f'(0)$ DNE since $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ DNE.